Weak Localization

Eric D. Black

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The 1977 Nobel Prize in Physics was awarded to Phillip Anderson, Sir Nevil Mott, and John van Vleck, *"for their fundamental theoretical investigations of the electronic structure of magnetic and disordered systems."* Their theoretical work stimulated a great deal of experimental activity that continues to this day, particularly in the area of disordered systems with restricted geometry. This lab will serve as an introduction to this field, as well as an introduction to low-noise measurement techniques used in many other condensed-matter fields.

Part I

1 Preliminary concepts

1.1 Nulled lock-in detection

Very often in physics, you have some sinusoidal signal whose amplitude is expected to change by small amounts, and you need to measure these small changes. (This will be the case in this lab.) One very effective way of doing this is by *nulling, i.e.* subtracting another signal at the same frequency and with nearly the same amplitude, then amplifying the difference between the two signals. If your experimental signal changes by only a few parts per million, it can be hard to resolve that change. However, if you want to measure a change by a factor of two in a signal 10^6 times smaller than your original signal, that is usually easier.

Consider two signals that have the same frequency and the same phase. In this case, the change in the difference is just equal to the change in the magnitude of one of the signals.

$$\begin{aligned} V_0(t) &= V_0 e^{i\omega t} \\ V_1(t) &= (V_1 + \delta V) e^{i\omega t} \end{aligned}$$

If $V_1 - V_0$ is small enough to be comparable to – or even smaller than – δV, then all you need to do is measure the amplitude of $V_1(t) - V_0(t)$ to get δV.

$$V_1(t) - V_0(t) = \delta V e^{i\omega t}$$

1. Consider the case where $V_1(t)$ has a slightly different phase from $V_0(t)$.

   $$\begin{aligned} V_0(t) &= V_0 e^{i\omega t} \\ V_1(t) &= (V_1 + \delta V) e^{i(\omega t + \phi)} \end{aligned}$$

   Sketch the complex amplitudes of $V_1(t)$ and $V_0(t)$ as vectors in the complex plane, and sketch the complex amplitude of the difference $V_1(t) - V_0(t)$. Assume $V_0 \approx V_1$ and $\phi \ll 1$. What are the amplitude and phase of the difference signal? How does the phase of the difference relate to the phase ϕ of $V_1(t)$?

   Hint: Since the factor of $e^{i\omega t}$ is common to both signals, and therefore to the difference between them, you can ignore it. This is the same as going to a rotating reference frame, where the real and imaginary axes rotate along with the signal. In this frame, V_0 always lies along the real axis.

2. Would measuring the change in the amplitude of $V_1(t) - V_0(t)$ give you δV? If not, what would?

1.2 Four-wire technique

The measurement leads of a sample in a cryogenic environment contain some resistance. There is an intimate relationship between electrical conductivity and thermal conductivity in a metal. If the wires leading down to the sample are good electrical conductors, they will also be good thermal conductors, and they will carry heat down from their room-temperature connections and heat up the sample. For this reason, most wires that connect a sample inside a cryostat to the outside world are made of some high-resistance material, such as constantan, an alloy of copper and nickel. This poses certain challenges to measuring the resistance of a cold sample, and a surprising amount of care is needed to get this kind of measurement right.

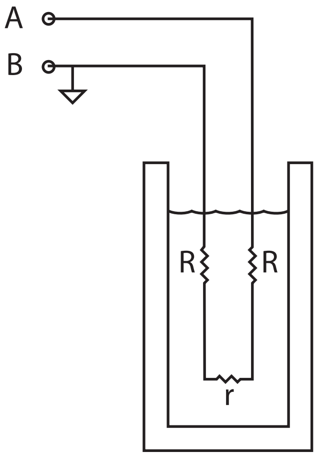

Figure 1: Measuring the resistance of a cold sample with resistive leads.

If we simply pass a current through the leads and measure the voltage across A and B, the resistance of the leads will contribute to that voltage. In Figure 1, we would measure a total resistance of $2R + r$, when we just want r. This would not be a problem, if we knew R to a high degree of precision, but usually we don't. First, R is usually much larger than r, and the voltage due to the leads overwhelms the signal due to the sample. Second, the large temperature gradient across the lead wires can lead to bizarre and unusual effects, such as thermoelectric potentials and weird magnetoresistances. If you want to study those effects, fine, but if you just want to study your sample r, then you'd better find a way to get rid of the effects of the lead wires.

You could use superconducting lead wires to get rid of R, but that would require cooling the leads, and you would still have a temperature gradient between the superconducting part and the room-temperature measurement electronics. You might also try running ordinary copper (high-conductivity) leads through the bath, and this is often done. However, the simplest and most robust way of isolating the voltage of the sample from that of the leads is to use two sets of lead wires, only one of which carries current. This is known as the four-wire technique, and it is illustrated in Figure 2.

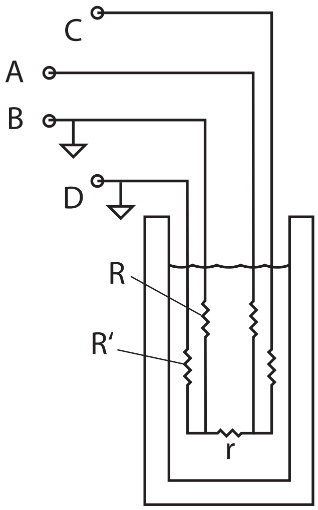

Figure 2: The four-wire measurement technique. Careful of your grounds!

Here, you pass a current through one set of leads and measure the voltage across the other. As long as no current flows through the voltage leads, their resistances, labeled R' in the figure, don't contribute to your signal. Now, this seems straightforward enough, and you may be wondering why I said this required a surprising amount of care earlier. The reason is that it is very difficult in practice to ensure that no current flows back through the voltage leads. Go back and look at Figure 2. Say you inject current at the point labeled "A." In a real experiment, "A" would be the inner conductor of some coaxial cable, and "B" would be the outer shield, held at ground. The voltage leads "C" and "D" would also be the inner and outer conductors of a coax cable, respectively. In the figure, it is fairly easy to see that current will flow back to both "B" and "D," because they are both at ground. In a real experiment, it is not always so obvious, and you have to find some way to break the ground connection at "D." When you do the lab you will learn two methods of breaking this *ground loop*, an obvious one using shielded connectors and a not-so-obvious one using isolation transformers.

3. If you apply a current I at the point "A" in Figure 2, what voltage will you measure at "C?" What voltage would you measure if you were to disconnect the ground at "D?"

1.3 Electrical conduction in films

The resistance of a rectangular block of metal with conductivity σ is given by

$$R = \frac{\ell}{wa\sigma}$$

where ℓ is the length of the sample in the direction of the current flow, and w and a are the width and height of the sample, respectively.

For a *square* sample, with $\ell = w$, the resistance only depends on the thickness.

$$R_{\Box} = \frac{1}{a\sigma}$$

The resistance of any sample can be expressed in terms of $R_{\Box}$ and the length to width ratio ℓ/w.

4. Do it. Express the length-to-width ratio as $n = \ell/w$. Yes, this is a dead simple exercise, but you will need this formula for the resistance when you analyze your data.

5. Using the bulk conductivity of silver, estimate the resistance of a film $100\mathring{A}$ thick, $5mm$ wide, and $25mm$ long.

6. If the conductivity changes by a small amount $\delta\sigma$, how does the resistance of a film change? Express the change in R in terms of the $R_{\Box}$ and $\delta\sigma$.

2 Experimental tasks 1: sample preparation

In the last lab, *Vacuum Techniques and Thin Film Deposition*, you grew a thin metal film that could be used as a sample for this lab. In this lab, you will cool a film down to $4.0K$ in the cryostat in Room 209, apply a magnetic field, and measure the resulting change in the film's resistance as a function of magnetic field.

It is best to use a relatively fresh sample for this lab. Older thin metal films tend to oxidize, and after a few weeks they will cease to conduct. If your film from Lab 3 has a resistance between about 25Ω and 250Ω, and you can make good electrical contact to it, you should use it for this lab. If not, don't worry about it. Such things happen in experimental physics. Just make another one.

Part II

3 Disorder and localization

In a perfectly periodic potential, electrons are free to move ballistically, without suffering any loss of momentum over time. (The theory of such *Bloch waves* is covered in any introductory solid-state physics text, and we won't go into it in detail here. If you want to learn solid state physics, Tanner [1] offers an excellent introduction to the subject. The standard encyclopedia of the subject is Ashcroft and Mermin [2], which is an excellent reference after you have read Tanner.) We know from x-ray diffraction experiments that metals exhibit periodic structure on a microscopic scale, and we also know that it is electrons that carry electrical current. Yet, when a current is set up in a metal, it usually dies out rather quickly unless there is a voltage present to push it along. Real electrons in a normal metal do not move ballistically for long.

The resistance to an electrical current that a normal metal has is due to disorder. No crystal lattice in nature is perfectly periodic. Vibrations due to heat distort the lattice at finite temperatures, and even at very low temperatures, some static disorder is usually present in form of dislocation, vacancies, or impurities. This static disorder would remain even if the system were at absolute zero. A theoretical model can be constructed where an electron moves ballistically between regions of disorder, where the potential is periodic, but gets scattered when it encounters an interruption in the periodicity of the lattice. A semiclassical treatment of this model, where electrons are considered point particles that obey Fermi-Dirac statistics, is quite simple and yields predictions that are in good agreement with nature.

A model of a disordered conductor that takes in to account the specifically wavelike properties of electrons is somewhat more complicated, and under normal circumstances, it is not necessary. The reason for this is that inelastic scattering (mostly from phonons) usually prevents an electron from retaining phase coherence over distances much longer than the mean free path. Coherent quantum mechanical interference cannot occur over longer distances, and a semiclassical picture applies. At low temperatures, however, this inelastic scattering is suppressed, and elastic scattering dominates, allowing electrons to maintain phase coherence over relatively large distances. This phase coherence gives rise to coherent interference phenomena which can lead to macroscopically observable effects. You will observe one of these effects, Weak Localization in a thin metal film, in this lab.

3.1 Behavior in one, two, and three dimensions

There is a theorem that, in a one-dimensional sample at zero temperature, any amount of disorder, no matter how small, will force the sample to become an insulator. The electrons get *localized* by any small deviation from perfect periodicity in a one-dimensional potential. This theorem is related to a similar one for plane waves, which you will prove in the exercises. In three dimensions, a large amount of disorder will also localize electrons, but a small amount will not. In two dimensions, the behavior is the same as in one. At absolute zero, any arbitrarily small amount of disorder leads to localization. This is somewhat surprising, since there is no corresponding simple theorem that guarantees the existence of bound states in two dimensions, as there is in one dimension.

3.2 Coherent backscatter I - Qualitative picture

The physical mechanism of localization due to disorder, whether in one, two, or three dimensions, is coherent backscattering. If one impurity scatters electrons isotropically, *i.e.* with equal probability in all directions, you might expect a random array of isotropic scatterers to also scatter electrons equally in all directions. In fact, they do not. If the electrons' wave functions can retain phase coherence over long distances, then a collection of isotropic scatterers produces more backward scattering than in any other direction. The degree to which reflection is enhanced depends on the distance over which electrons can retain phase coherence. In two dimensions, if the phase-coherent distance is infinite, then the backscattering probability becomes one, and the electrons are localized.

To understand how phase coherence leads to enhanced backscattering, consider an array of point scatterers, as shown in Figure 3. An electron's wave function enters this array and scatters isotropically off of every point defect it encounters. Scattered waves will also scatter when they encounter defects in the lattice, leading to second order scattered waves, third order waves, and

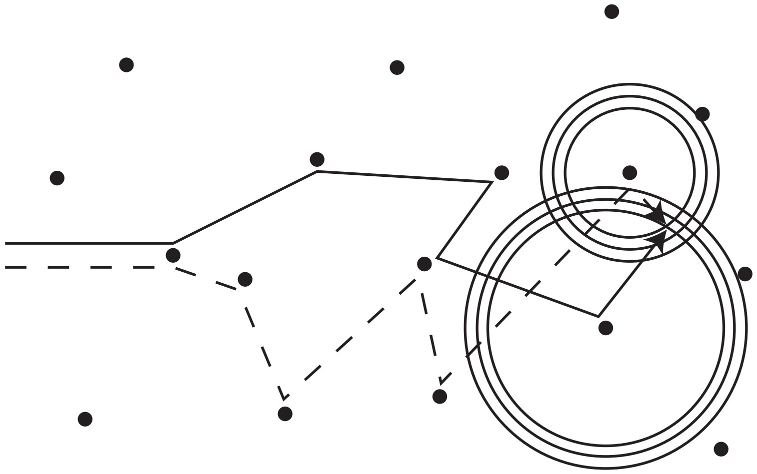

Figure 3: A random array of isotropic scatterers.

so on. The net wave function is the sum of all of these multiply-scattered waves.

If we take two randomly chosen waves and look at their sum, their phase relation will not, on average, be anything special. These two waves are just as likely to interfere destructively as constructively, and on average they will add incoherently. For any random direction, the net scattering probability is just the incoherent sum of all of the individual waves.

Consider now a specially-chosen pair of waves, as shown in Figure 4. These traverse the same nearly closed loop in opposite directions. One, represented by the solid line, travels around the loop in a counterclockwise direction. The other, represented by the dashed line, traverses the loop in a clockwise direction. The beginning of the path enclosed by the solid line corresponds to the end of the path enclosed by the dashed line and *vice-versa.* If the beginning and end points of the loop are aligned with the direction of the incoming waves, there exists a region where the outgoing waves are always in phase. The reason for this phase coherence is simple – the optical path lengths are the same around the loop regardless of the direction travelled.

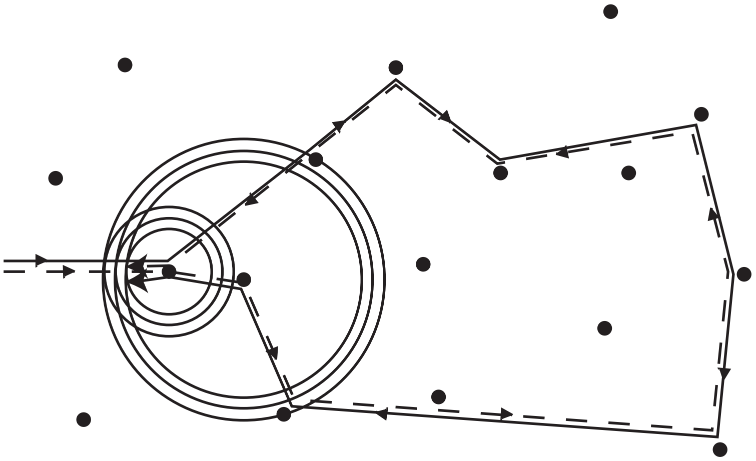

Figure 4: For every loop that returns near the origin, there is another one going around the same path backwards that has the same phase.

This phase coherence is found only in a cone directed in the opposite direction of the incoming wave, and it occurs only if the end points of the loop are lined up with a given accuracy. This coherent interference will be present for all pairs of complimentary scattering paths with end points close and aligned, and all these pairs will add together to produce a significant backscattered "bright spot" in addition to that produced by isotropic scattering. This increased probability of reflection (over the isotropic case) must translate to a decreased probability of transmission, with implies a decreased conductivity for the sample.

7. *Optional:* Show that every attractive potential, no matter how shallow, in one dimension has at least one bound state.

   Hint: Assume that the attractive potential is negative everywhere and goes to zero as x goes to infinity in both directions. First show that the energy of any state cannot be any smaller than the lowest-energy eigenvalue of the hamiltonian of the system. Then show that it is possible to pick a Gaussian wave function that has a negative expectation value for its energy. Put the two together to show that the lowest-energy eigenstate must have negative energy, and therefore be bound.

3.3 Experimental tasks 2: electronics

Once you have a sample that you can use, install it in the cryostat, and measure its resistance using the four-wire technique we talked about previously. Do this at room temperature. Don't bother with liquid helium at this stage, and don't worry about nulling yet. For a current source, use the lock-in's internal oscillator in series with a large, stable resistor. This *load resistor* should have a resistance that is substantially greater than the sample's, so that changes in the sample's resistance won't have much effect on the excitation current. Beware of ground loops! You may want to insert a 1:1 transformer between the lock-in's voltage source and the load resistor to prevent them.

Now null the signal. To generate $V_0(t)$, split the excitation voltage, and send one copy of it to the load-resistor/sample circuit and the other to the input of the DT72A inductive divider. The DT72A is just a low-noise variable transformer. The output voltage is some fraction of the input voltage, and you get to adjust that fraction. Each knob on the front of the box controls one digit in that fraction. For example, you can dial in $V_{out}/V_{in} = 0.1415926$ by setting the first knob to 1, the second to 4, the third to 1, etc. Send the output of the DT72A to the B-input of the lock-in, and measure the difference between the sample voltage and the DT72A's output voltage. Adjust the DT72A until this difference is as small as possible.

While you *can* use the digital readout of the lock-in to take data, you can often get a much better feel for what's going on by sending the lock-in's output voltage to an oscilloscope and using the scope to measure the changes in this voltage. Full scale on the digital readout should correspond to $10V$ on the output voltage, so the gain of the lock in is just $10V$ divided by its sensitivity.

3.4 Experimental tasks 3: anticipating the signal size

Calculate the change you would expect in the sample voltage if the sample's conductance ($a\delta\sigma$) were to change by a single conductance quantum e^2/h. This will be the order of magnitude of your weak-localization signal. Calculate the voltage change you expect to see at the output of the lock-in, and compare it with the level of noise you see in your measurement. Do you have the resolution to see weak-localization magnetoresistance in your sample? If not, what can you change in your electronic setup to achieve it?

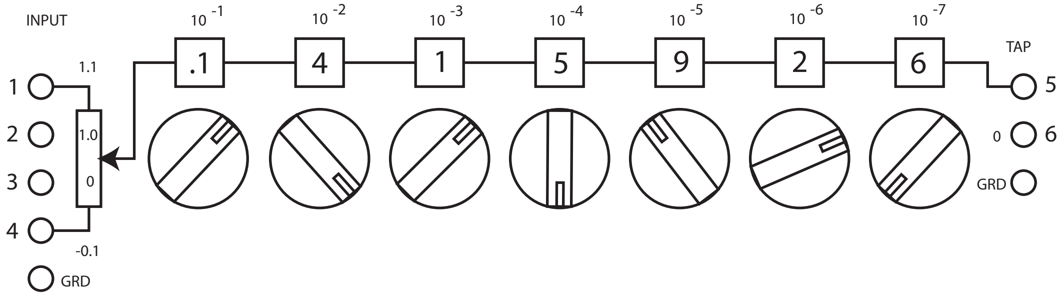

Figure 5: The front face of the DT72A variable transformer. Sockets 2 and 3 are the normal inputs, and sockets 5 and 6 tap the output (the secondary coil of the transformer). Use the knobs to dial in the fraction of the input voltage you want to appear at the output. The sockets are bannana-type, so you will have to use BNC-to-bananna adapters at both the input and the tap.

If, for some reason, you decide that you just can't get the resolution you need from your existing setup, you can amplify the sample voltage using a low-noise preamp. This is especially useful if your sample resistance is small, and we have a Stanford Research SR560 in the lab for this purpose. If you decide to use a preamp, don't forget to include its gain when calculating the change in the sample voltage $\Delta V(B)$ when you apply a magnetic field.

Part III

4 Diffusion

4.1 Fick's law

A classical particle bouncing around in a random collection of scatterers executes a random walk. Because of the wandering nature of the path it takes, the net distance it travels after many bounces is not linear in time. The mean-square distance L a particle travels after a random walk of time t

is

$$< L^2 >= 2v_F\ell t \tag{1}$$

where ℓ is the mean free path of the particle, and v_F is its velocity (the Fermi velocity for semiclassical conduction electrons). Many independent particles undergoing random walks obey *Fick's law*:

$$\vec{J} = -D\vec{\nabla}\phi \tag{2}$$

where ϕ is the number of particles per unit volume, and $\vec{J}$ is the current density of particles. D is known as the Diffusion constant, and in terms of the microscopic properties of the random walk, it is

$$D = \frac{1}{n}v_F\ell \tag{3}$$

where n is the number of dimensions the particles are free to move in. Fick's law, along with a conservation law for the number of particles, leads in a straightforward way to the diffusion equation.

$$\frac{\partial\phi}{\partial t} = D\nabla^2\phi \tag{4}$$

8. *Optional:* Derive Equation 1, and show how Fick's law follows from it. Show where the expression for D (Equation 3) comes from.

9. *Optional:* Implicit in Equation 2 is the assumption that the total number of particles is conserved. What form would Equation 2 take if particles were being injected into - or removed from - the system?

4.2 Green's functions

Green's functions are widely used in condensed-matter theory. In simplest terms, a Green's function is the response of a system to a point source. Say we have some system whose behavior is governed by an inhomogeneous differential equation.

$$\mathcal{L}[\phi(\vec{r},t)] = \rho(\vec{r},t) \tag{5}$$

where $\mathcal{L}$ represents an operator, $\rho(\vec{r},t)$ represents a source, and we want to find the function $\phi(\vec{r},t)$ that describes the behavior of the system. A common example is electrostatics, where $\mathcal{L} = \nabla^2$, ρ is the charge, and ϕ is the potential.

Now, most *homogeneous* differential equations have been well studied, and you can look up solutions to them or have Mathematica tell you what they are. The difficulty, in our electrostatics example and in most other physics problems, comes from the source term $\rho(\vec{r}, t)$, which may be different for every problem. What we need, then, is a general solution to the inhomogeneous problem and a way to connect it to a specific source $\rho(\vec{r}, t)$. Green's functions provide that solution.

The Green's function $G(\vec{r}, t; \vec{r}_0, t_0)$ associated with a particular differential operator $\mathcal{L}$ is the solution to a very specific inhomogeneous differential equation. It is the solution when the source $\rho(\vec{r}, t)$ is a point, or delta function, at the coordinate $\vec{r}_0$ and, where appropriate, occurring at time t_0.

$$\mathcal{L}[G(\vec{r}, t; \vec{r}_0, t_0)] = \delta^{(n)}(\vec{r} - \vec{r}_0)\delta(t - t_0) \tag{6}$$

Another way of saying this is that $G(\vec{r}, t; \vec{r}_0, t_0)$ is the response of the system to a point source. In our electrostatics example, the point source would be a point charge, and the Green's function is the familiar $1/r$ potential for a point charge. (Note that in this case there is no time dependence, and we just leave out the $\delta(t - t_0)$.) In the same way that any arbitrary source, no matter what its shape, can be thought of as a collection of point sources,

$$\rho(\vec{r}, t) = \int \rho(\vec{r}_0, t_0)\delta(\vec{r} - \vec{r}_0)\delta(t - t_0) d^n r_0 dt_0$$

the response of the system can be found by summing up the responses from all the individual points that make up the source.

$$\phi(\vec{r}, t) = \int G(\vec{r}, t; \vec{r}_0, t_0)\rho(\vec{r}_0, t_0) d^n r_0 dt_0 \tag{7}$$

Note that this method only works for linear differential equations, in the sense that $\mathcal{L}[A] = \alpha$ and $\mathcal{L}[B] = \beta$ imply $\mathcal{L}[A+B] = \alpha + \beta$. Physically, this means that the system has to obey linear superposition: The response of the system to two or more sources is just the sum of the individual responses. Linear superposition holds for electric fields and potentials, quantum-mechanical wave functions that are solutions to the Schrödinger equation, and just about anything else we are usually interested in.

In weak-localization theory, we will deal with the diffusion equation a lot because it describes the transport of an electron through a random array of scatterers.

10. Show that the solution $\phi(\vec{r}, t)$ given by Equation 7 is a solution to Equation 5 for any linear operator $\mathcal{L}$.

The following problems involve the normalized Green's function for the diffusion equation.

$$G(\vec{r}, t; \vec{r}_0, t_0) = \begin{cases} \frac{\exp[-|\vec{r}-\vec{r}_0|^2/4D(t-t_0)]}{(4\pi D(t-t_0))^{n/2}} & \text{if } t \geq t_0, \\ 0 & \text{if } t < t_0. \end{cases} \tag{8}$$

where n is the number of dimensions, and D is the diffusion constant. The physical meaning of this Green's function is this: $G(\vec{r}, t; \vec{r}_0, t_0)$ represents the probability density for finding a particle as it undergoes a random walk. In the case of weak localization, the particle will be a conducting electron that is bouncing off scattering centers - *i.e.* diffusing - through a disordered solid. The probability dP of finding a particle that started out at $\vec{r_0}$ at time $t = t_0$ in a small volume $dV = d^n\vec{r}$ around the point $\vec{r}$ is

$$dP = G(\vec{r}, t; \vec{r_0}, t_0)dV$$

11. For the function $G(\vec{r}, t; \vec{r}_0, t_0)$ defined in Equation 8, show that normalization is preserved at all times $t > t_0$, *i.e.* show that the probability of finding a particle anywhere is 1, regardless of when you look for it.

12. Show that G is the Green's function for the diffusion equation, *i.e.* show that it satisfies

$$\left[\frac{\partial}{\partial t} - D\nabla^2\right] G(\vec{r}, t; \vec{r}_0, t_0) = \delta(t - t_0)\delta^{(n)}(\vec{r} - \vec{r}_0)$$

where n is the number of dimensions, D is the diffusion constant, and $t > t_0$. For time $t < t_0$, $G(\vec{r}, t; \vec{r_0}, t_0) = 0$.

13. Now consider boundary conditions. What would G become if diffusion were limited to a slab of thickness $a \gg \ell$ around the x-y plane? Assume that the particles are confined to $-a/2 < z < +a/2$, but that they are free to move about in the x and y directions. Write down an expression for G good for times $t \gg D/a^2$, *i.e.* when the particles have spread out in the z-direction as much as they are going to.

    This kind of sample is called *quasi*-two-dimensional, because it is essentially two-dimensional for diffusion, at least for times $t \gg D/a^2$, but microscopically the motion occurs in three dimensions, $a \gg \ell$. There are whole research fields built around studying such *mesoscopic* systems, physical objects that exhibit behavior that is microscopic in one sense and macroscopic in another, or even behavior that is neither truly microscopic nor macroscopic. Quasi-two-dimensional samples are relatively easy to make, and in fact you did make one when you grew your thin metal film in the vacuum lab. Weak localization is present in samples of any dimensionality, but it is easiest to see in quasi-two-dimensional samples.

Electrons are conserved. They don't just vanish, so the probability density G ought to be conserved as well. They may, however, lose their phase coherence, and in weak localization it is only those electrons that retain their phase coherence that we are interested in. To describe these electrons, we need a special kind of Green's function called a *Cooperon*, which, in this case, is a decaying Green's function for the diffusion equation.

$$W(\vec{r}, t; \vec{r}_0, t_0) = G(\vec{r}, t; \vec{r}_0, t_0) e^{-(t-t_0)/\tau_\phi}$$

$W(\vec{r}, t; \vec{r}_0, t_0)$ is the Cooperon. (The "Coop" part is pronounced as in "chicken coop." More appropriately, it is pronounced as in the last name of Leon Cooper, of Bardeen-Cooper-Schrieffer fame, winner of the 1972 Nobel Prize in Physics and inventor of the Cooperon propagator W.) All we've done here is to multiply two probabilities. G is the probability of finding an electron, and $\exp(-(t-t_0)/\tau_\phi)$ is the probability that the electron has not lost its phase coherence in time $(t-t_0)$. One of the main ways phase coherence is destroyed for real electrons in a real sample is when the electron collides with a phonon. These collisions can pretty much always happen when the temperature of the sample is not zero. The quantity τ_ϕ is the average time to an inelastic scattering event, and it should not be confused with τ, with is the time to an elastic scattering event (one that preserves memory of the electron's phase). As the temperature is lowered, phase-destroying events ought to become rarer, and τ_ϕ ought to increase.

4.3 Experimental tasks 4: cryogenic measurements

Get your T.A. to help you transfer liquid helium into the cryostat, and cool your sample down to 4.0K. Measure the sample's resistance as a function of temperature as you cool it. The resistance should change by at least a factor of two (and probably more) between room temperature and 4.0K. There should be no need to null for this measurement. Can you explain the $R(T)$ curve you observe?

Now null the signal and send the output of the lock-in to an oscilloscope, just as you did at room temperature when you set up your electronics the first time. Increase the gain of the lock-in and make any other adjustments you need to resolve the weak-localization magnetoresistance. Apply a magnetic field, and look for magnetoresistance. When you see it, check to make sure that it follows the same curve as you ramp the field up and then back down again. Also check negative field values, since the weak-localization magnetoresistance should only depend on the absolute value of B, not its sign.

Once you feel confident you are seeing weak-localization magnetoresistance, record $\Delta R(B)$ for several field values. Use enough points to resolve the shape of the curve, since you will be fitting it to weak-localization theory later.

Optional: Measure the magnetoresistance curve at several different temperatures. What happens at temperatures below $4.0K$? How warm can the sample get before it no longer exhibits a magnetoresistance? Does the magnetoresistance have the same basic shape at all temperatures, or are there regions where its qualitative character is fundamentally different?

Part IV

5 Conductivity of a disordered metal

5.1 The Einstein relation

The Einstein relation gives the conductivity $\sigma(\omega)$ of a metal as a function of the frequency of an applied voltage $\omega = 2\pi f$.

$$\sigma(\omega) = e^2 g(E_F) D(\omega),$$

where $g(E_F)$ is the density of states per unit volume at the Fermi surface, and $D(\omega)$ is the diffusion constant in the frequency domain. In the time domain, this diffusion constant is just a velocity correlation function

$$D(t) = \frac{1}{n} \langle \vec{v}(t) \cdot \vec{v}(0) \rangle$$

where n is the number of dimensions the electrons are free to move about in (3 in bulk), $\vec{v}(0)$ is the velocity of an electron at time zero, and $\vec{v}(t)$ is the velocity of the same electron after time t has elapsed. For our purposes, we can assume that the electrons travel ballistically between scattering events, so that, if no scattering event occurs during the time t,

$$\vec{v}(t) \cdot \vec{v}(0) = v_F^2$$

where v_F is the Fermi velocity. If a scattering event does occur,

$$\vec{v}(t) \cdot \vec{v}(0) = 0$$

On average, then, if the scattering time is τ

$$\langle \vec{v}(t) \cdot \vec{v}(0) \rangle = v_F^2 e^{-t/\tau}$$

14. Using these assumptions, show that the conductivity is

    $$\sigma(\omega) = \frac{Ne^2\tau}{m} \frac{1}{1 - i\omega\tau},$$

    where N is the number of conducting electrons per unit volume, e is the charge of an electron, and m is its mass.

    Hint 1: Treat the conducting electrons as a gas of free particles. With this assumption, you can use the free-electron value for the density of states:

    $$g(E) = \frac{3}{2} \frac{N}{E_F} \sqrt{\frac{E}{E_F}}$$

    This is the assumption Drude used when he first calculated the conductivity of a metal. This result, known as the Drude conductivity, is a pretty good approximation for most metals.

    Hint 2: $D(t)$ doesn't have any physical meaning for $t < 0$ in our picture, and you can treat it as zero for negative times. This means your limits of integration in your Fourier transform should be from zero to infinity in t, rather than from negative infinity to positive infinity.

    $$D(\omega) = \int_0^\infty e^{i\omega t} D(t) dt$$

5.2 Coherent backscatter II - Quantitative prediction

Coherent backscattering gives rise to a correction to the velocity correlation function, which becomes

$$\langle \vec{v}(t) \cdot \vec{v}(0) \rangle = v_F^2 \left(e^{-t/\tau} - P_r(t) \right)$$

where the correction term

$$P_r(t) = \frac{1}{2\pi} W(\vec{0}, t; \vec{0}, 0) \ell^3 \left(\frac{\lambda_F}{\ell} \right)^2$$

is small compared to the Drude part $e^{-t/\tau}$. In this expression, $W(\vec{r}, t; \vec{0}, 0)$ is the Cooperon describing the probability density that, if an electron starts at the origin $\vec{0}$ at time 0, it will be found at $\vec{r}$ at a later time t with its phase coherence intact. $W(\vec{0}, t; \vec{0}, 0)\ell^3$ is the probability that an electron will diffuse back into a volume ℓ^3 centered around where it started at $t = 0$ and retain its phase coherence.

It can be shown that the angular size of the phase coherent spot is, in the limit $\lambda_F/\ell \ll 1$ (weak disorder),

$$\delta\theta \approx \sqrt{\frac{\lambda_F}{\ell}}$$

The probability of an electron scattering into this phase coherent spot (in addition to the probability of being isotropically scattered) would be just the ratio of the solid angle covered by the spot to 4π, the total solid angle available for scattering. This may be approximated as just

$$\delta\theta^2 \approx \left(\frac{\lambda_F}{\ell} \right)$$

Another factor of λ_F/ℓ is included to account for the probability of having the beginning and end points of the scattering path lined up with the direction of the incoming electrons' velocity within a factor of $\delta\theta$, which is also required for a phase coherent spot to exist. The factor of $1/2\pi$ is the result of a rigorous, quantitative analysis using quantum field theory [3].

15. Evaluate the correction to the conductivity due to coherent backscattering at $\omega = 0$, for a quasi-two-dimensional sample of thickness a. Express your answer in terms of the universal conductance quantum

$$\frac{e^2}{h} = \frac{1}{26k\Omega}$$

Hint 1: Only those electrons that travel out around some closed loop and then return to the origin contribute to the coherent backscatter correction. You will need to neglect the behavior of the correction for short times ($t < \tau$), when the electrons have not yet left the origin. This means that the limits of integration in your Fourier transform should be from τ to infinity, rather than from zero to infinity. Unfortunately, this makes for a real ugly integral. The trick to evaluating it is due to Chakravary and Schmid [4], who make the approximation

$$\int_{\tau}^{\infty} \frac{\exp\left(-\left(\frac{1}{\tau_\phi} - i\omega\right)t\right)}{t} dt \approx \int_{0}^{\infty} \frac{\exp\left(-\left(\frac{1}{\tau_\phi} - i\omega\right)t\right) - \exp\left(-\frac{t}{\tau}\right)}{t} dt$$

Hint 2: There is a relationship between the number density and the Fermi wave number k_F that you might find useful. It is

$$N = \frac{k_F^3}{3\pi^2}$$

This, as well as the expression for $g(E)$ in terms of the number density and the Fermi energy, is derived in Ashcroft and Mermin [2].

16. *Optional:* The inelastic dephasing time τ_ϕ is expected to be related to temperature by a power law. Assuming $\tau_\phi = A \times (T_0/T)^p$, where A and p are constants, and T_0 is some characteristic temperature, show that the resistance of a thin film varies linearly with the log of the temperature. If you were to measure R as a function of T and do a linear fit between R and $\ln(T/T_0)$, what would the fit parameters (slope and y-intercept) tell you about the sample?

    Detailed microscopic theories of electron and phonon transport give predictions for the constant p. It often happens that theorists will come up with several different theories, each of which predicts a different value for p. When this happens, experimentalists step in, measure p, and more often than not declare that the actual value is something completely unexpected, and that all of the theories were wrong.

One of the first predictions of weak-localization theory was that the resistance of a thin film should increase logarithmically as the temperature is lowered. Initially, it was thought that this would be a definitive signature of weak localization. A number of people immediately set out to measure the resistance versus temperature, and they did, if fact, see a clear logarithmic dependence, just as predicted [5, 6, 7]. However, it was quickly discovered that electron-electron Coulomb interactions could also produce the same effect [8]. The temperature dependence of the conductivity correction does not give conclusive evidence of weak localization. What we need, if we want to study this effect, is a way of turning the coherent backscattering on or off without affecting anything else. If you are familiar with the Aharonov-Bohm effect, you will see where this is going.

Think back to the complimentary closed loops in Figure 4. The electron goes the same distance whether it travels clockwise or counterclockwise. Phase coherence is preserved between the two loops because they are the same length, as long as the only thing that contributes to the phase is the distance traveled. If we introduce a magnetic field perpendicular to the page, it will contribute a phase to each path that is proportional to the flux enclosed by the loop. However, this phase will not be the same for both loops! The phase from the flux enclosed will be positive for one direction and negative for the other, and the symmetry between the two paths will be broken. This destroys phase coherence and turns off the weak-localization correction to the conductivity. The resulting magnetoresistance, the change in the resistance of a film as a magnetic field is applied, is what you will observe in the lab.

5.3 Magnetic field

In the presence of a magnetic field, the diffusion equation for charged particles becomes

$$\left[\frac{\partial}{\partial t} + D\left(i\nabla + \frac{2e}{\hbar}\vec{A}(\vec{r})\right)^2\right] G(\vec{r}, t; \vec{0}, 0) = \delta(t)\delta^{(n)}(\vec{r})$$

where $\vec{A}(\vec{r})$ is the vector potential. If the sample is confined to the xy plane, and the field is along the z axis, we may choose

$$\vec{A}(\vec{r}) = Bx\hat{\jmath}$$

The Green's function in this case becomes, at the origin,

$$G(\vec{0}, t; \vec{0}, 0) = \left(\frac{2eB}{ha}\right)\sum_n \exp\left(-2\frac{eDB}{\hbar}(2n+1)t\right)$$

The conductivity correction is evaluated by the same procedure you used in the exercise, and the result is

$$\delta\sigma(B) = -\frac{1}{a}\frac{e^2}{\pi h}\sum_n \left\{ \left[n + \frac{1}{2} + \frac{\hbar}{4eBD\tau_\phi} \right]^{-1} - \left[n + \frac{1}{2} + \frac{\hbar}{4eBD\tau} \right]^{-1} \right\}$$

This doesn't look very enlightening at first, but it turns out that the sum can be evaluated in terms of the *digamma function*, the logarithmic derivative of the gamma function.

$$\psi(z) = \frac{d\,[\ln\Gamma(z)]}{dz} = \frac{\Gamma'(z)}{\Gamma(z)} \tag{9}$$

where $\Gamma(z)$ is the usual Gamma function. There are many definitions for the Gamma function. The one most commonly used in physics, and the one that will be of most use to us, is

$$\Gamma(z) = \int_0^\infty t^{(z-1)} e^{-t} dt \tag{10}$$

where $\Re\{z\} > 0$.

With the sum evaluated in terms of ψ, the conductivity becomes

$$\delta\sigma(B) = \frac{1}{a}\frac{e^2}{\pi h}\left\{ \psi\left(\frac{1}{2} + \frac{\hbar}{4eBD\tau_\phi}\right) - \psi\left(\frac{1}{2} + \frac{\hbar}{4eBD\tau}\right) \right\}$$

5.4 Digamma function

17. *Optional:* Using the definitions of the Gamma and Digamma functions (Equations 10 and 9), and assuming z is always real and positive, show the following.

    (a) $\Gamma(z+1) = z\Gamma(z)$

    (b) $\Gamma(1) = 1$

    (c) From the two results above, convince yourself that $\Gamma(z) = (z-1)!$ when z is a positive integer.

    (d) $\Gamma(z) \approx -\gamma + (1/z)$ as $z \to 0$, where γ is Euler's constant.

(e) Sketch a graph of $\Gamma(z)$. Don't have a computer or graphing calculator do it for you! This is an optional exercise, after all. Instead, reason out what it should look like and why, then draw a picture of what you expect.

(f) $\psi(z+1) = \psi(z) + (1/z)$

(g) $\psi(z) \approx -\gamma - (1/z)$ as $z \to 0$.

(h) $\psi(z) \approx \ln(z) - (1/2z) - (1/12z^2) + \cdots$ when $z \gg 1$. For this one, you may want to use *Stirling's approximation*,

$$N! \approx \sqrt{2\pi N} N^N \exp\left(-N + \frac{1}{12N} + \cdots\right)$$

(i) Sketch a graph of $\psi(z)$.

18. The elastic scattering time τ is usually very short, compared with the inelastic time τ_ϕ. We go to low temperatures precisely to obtain this condition, which allows us to observe weak localization. In fact, for all practical laboratory magnetic fields B, τ is usually so small that

$$B \ll \frac{\hbar}{4eD\tau}$$

The inelastic dephasing time τ_ϕ, however, is large enough at low temperatures that our laboratory B-field might be larger or smaller than the characteristic field $\hbar/4eD\tau_\phi$. Using the approximation, for large z

$$\psi(z) \approx \ln z - \frac{1}{2z} - \frac{1}{12z^2} + \cdots$$

show that the magnetoresistance does not depend on τ, and calculate the expected change in resistance of a thin film $\delta R(B) = R(B) - R(0)$ as a function of magnetic field B, the resistance per square of the film $R_\square$, the total resistance of the film $R(0)$, and the inelastic scattering length $L_\phi^2 \equiv D\tau_\phi$. Plot this *magnetoresistance*, with appropriate units on the y-axis, for $L_\phi = 0.5\mu\text{m}$. Over what range would you need to scan the magnetic field to best see the weak localization effect?

5.5 Spin effects

We have seen two mechanisms that can break the symmetry between oppositely-oriented closed loops: an applied magnetic field B, and random, usually thermal effects lumped together in τ_ϕ. There is a third category that we must discuss if we are to analyze data from a real film, and that is due to electron spin. So far, we have not discussed spin, but for two real electrons to remain in phase after a series of scattering events, their spin states must also stay in phase. There are many things that can affect the spin of a conducting electron, the two most prominent being spin-orbit and spin-flip interactions. Spin-orbit coupling is, just as in the elementary hydrogen atom, an interaction between the orbital angular momentum of the electron and its spin. Orbital angular momentum, in this case, is between the electron and a scattering center, and it is present even though the electron is not bound to the scatterer. Angular momentum must be conserved, but it can be passed between orbital and spin degrees of freedom. When this happens, the phase of the spin state is altered, and this will affect the coherent backscattering pattern.

Spin-flip scattering occurs when there is an exchange of angular momentum between the scattering center and the electron itself, without orbital angular momentum being involved. While it is possible to construct samples that show evidence of spin-flip scattering, most of the time you don't see it. It's not the kind of thing you stumble across by accident.

Both spin-orbit and spin-flip effects can be taken into account by including an additional term in the correction $P_r(t)$.

$$P_r(t) = \frac{1}{2\pi} W(\vec{0}, t; \vec{0}, 0)\ell^3 \left(\frac{\lambda_F}{\ell}\right)^2 C_s(t)$$

where the additional term $C_s(t)$ is known as the spin Cooperon. Its derivation is beyond the scope of this handout, but it has been evaluated by theorists to be [4]

$$C_s(t) = \frac{3}{2} \exp\left[-\left(\frac{4}{3\tau_{so}} + \frac{2}{3\tau_{sf}}\right)t\right] - \frac{1}{2} \exp\left[-\frac{2}{\tau_{sf}}t\right]$$

where τ_{so} is the spin-orbit scattering time, and τ_{sf} is the spin-flip scattering time. Evaluating the conductivity correction with the spin Cooperon gives

$$\begin{aligned}\delta\sigma(B)-\delta\sigma(0) &= \frac{1}{a}\frac{e^2}{\pi h}\left\{\frac{3}{2}\left[\psi\left(\frac{1}{2}+\frac{\hbar}{4e}\frac{1}{L_1^2B}\right)-\ln\left(\frac{\hbar}{4e}\frac{1}{L_1^2B}\right)\right]\right.\\ &\left.-\frac{1}{2}\left[\psi\left(\frac{1}{2}+\frac{\hbar}{4e}\frac{1}{L_0^2B}\right)-\ln\left(\frac{\hbar}{4e}\frac{1}{L_0^2B}\right)\right]\right\}\end{aligned}$$

where L_0 and L_1 are the singlet and triplet dephasing lengths, and they are given by

$$\frac{1}{L_0^2}=\left(\frac{1}{L_\phi^2}+\frac{2}{3L_{sf}^2}\right)+\frac{4}{3L_{sf}^2}$$

and

$$\frac{1}{L_1^2}=\left(\frac{1}{L_\phi^2}+\frac{2}{3L_{sf}^2}\right)+\frac{4}{3L_{so}^2}$$

The inclusion of spin effects can lead to some bizarre and unexpected results, including a reversal of the sign of the correction to the conductivity. If spin effects are strong enough, particularly spin-orbit scattering, coherent backscattering will lead to weak *anti*-localization.

19. Plot the expected magnetoresistance, in appropriate units, with $L_0 = 0.5\mu\text{m}$ and $L_1 = 0.5\mu\text{m}$, $L_1 = 0.2\mu\text{m}$, $L_1 = 0.1\mu\text{m}$, $L_1 = 0.03\mu\text{m}$, and $L_1 = 0.01\mu\text{m}$. Over what range would you need to scan the magnetic field to best see the weak localization effect?

20. *Optional:* Using the appropriate asymptotic behavior of $\psi(z)$, show that for small magnetic fields ($B \ll \hbar/4eL_0^2$ and $B \ll \hbar/4eL_1^2$), spin effects will lead to weak anti-localization when $L_1 < L_0/\sqrt{3}$.

21. *Optional:* Assume that spin-flip scattering is negligible, *i.e.* L_{sf} is infinite, spin-orbit scattering is constant, $L_{so} = 2.0\mu\text{m}$, and that inelastic dephasing obeys a power law with respect to temperature, $L_\phi = (1.0\mu\text{m})(4.0K/T)^2$. Plot L_0 and L_1 versus temperature from $1K$ to $20K$ on a log-log scale. Plot the weak-localization magnetoconductance at $1.0K$, $2.2K$, $4.0K$, and $20.0K$.

Spin-orbit scattering usually scales as Z^4, where Z is the atomic number of the material the film is made out of, so light metals should show orbital weak localization, whereas heavier metals should show spin effects. In this lab, you can observe spinless weak localization in magnesium films, and spin-orbit-dominated weak localization in silver films.

5.6 Experimental tasks 5: data analysis

See if you can fit the weak-localization theory to the data you collected, and extract values for L_0 and L_1, or L_ϕ, depending on which formula you use.

Optional: If you took data at several temperatures, do a fit for each, and plot the dephasing lengths versus temperature. Can you explain any of the features of your plot?

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